

Information metric on instanton moduli spaces in nonlinear σ models

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We study the information metric on instanton moduli spaces in two-dimensional nonlinear σ models. In the CP^1 model, the information metric on the moduli space of one instanton with the topological charge $Q = k (k \geq 1)$ is a three-dimensional hyperbolic metric, which corresponds to Euclidean anti-de Sitter space-time metric in three dimensions, and the overall scale factor of the information metric is $4k^2/3$; this means that the sectional curvature is $-3/4k^2$. We also calculate the information metric in the CP^2 model.

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I. INTRODUCTION

A parametrized family of probability distributions is often treated as a statistical model, which is deeply related to nonlinear σ ($NL\sigma$) models. The $NL\sigma$ models have arisen in various contents [1–3]: the Heisenberg ferromagnet model, the quantum Hall effect, and other statistical mechanics problems; and the conformal field theory, string theory, and Yang-Mills (YM) theories in four dimensions. In such models, instantons play an important role in nonperturbative analyses [4–6]. From the geometrical point of view, the information metric arises as a metric on the moduli space of the instantons, and in more general as a metric on a manifold of probability distributions. It is generally defined by

$$G_{AB} = \int d^D x p(x; \theta) \partial_A \ln p(x; \theta) \partial_B \ln p(x; \theta), \quad (1)$$

where $p(x; \theta)$ is the probability density function of x , parametrized by θ [7,8]. Here, x is assumed to belong to a flat D -dimensional space \mathbf{R}^D , θ is a real N -dimensional parameter $(\theta_1, \theta_2, \dots, \theta_N)$, and ∂_A is the derivative with respect to the parameters θ_A ($A = 1, 2, \dots, N$). The space of the parameters corresponds to the instanton moduli space in this paper. If we treat the normalized topological charge density of the instantons, which is the same as the normalized energy density, as the probability density function, the information metric on the moduli space of the instantons naturally appears from the definition (1).

Geometrical approaches have many advantages in statistical physics [9–11], for example, it has recently been shown that the scalar curvature of the information metric plays a central role in studying the characterization of the phase structure in statistical mechanics models [12–15]. In this paper, we concentrate upon the geometrical perspective from which instantons in nonlinear σ models are combined with the information metric. The information metric on the moduli space of the instantons has been studied in YM theories in four dimensions and in $NL\sigma$ models in two dimensions. It has been shown so far that the information metric is isometric to hyperbolic space when one instanton with topological charge 1 exists in a $SU(2)$ YM theory [16–18] or in a

rational map [19] which is correspondent to a $NL\sigma$ model in two dimensions, and also that the information metric is non-degenerate for the moduli space of the multiinstantons in rational maps [19]. What has not been done so far is to study the concrete dependence of the information metric on the topological charge of the instantons. That is what we shall do in this paper. We shall, in particular, compute the overall scale factor, which corresponds to the square of “radius,” of the information metric in the moduli space of one instanton with the topological charge which is greater than or equal to 1. The negative reciprocal of the overall scale factor of the information metric is directly correspondent to the curvature, which is the sectional curvature to be exact. In the geometrical analyses about the $NL\sigma$ models and statistical models in general, it is important to investigate the curvature of the moduli spaces where the instantons live.

The paper is organized as follows. In Sec. II, we briefly review the CP^n $NL\sigma$ model. In Sec. III, we compute the information metric on the moduli space of one instanton with the topological charge which is greater than or equal to 1 in the CP^1 model. And, furthermore, we compute the information metric in the CP^2 model. In Sec. IV, we draw conclusions and indicate some interesting possibilities to extend our work.

II. DEFINITION OF THE CP^n $NL\sigma$ MODEL

To define the two-dimensional CP^n model, we take an $(n+1)$ -dimensional complex vector field Φ [20,21],

$$\Phi = (\phi_1, \phi_2, \dots, \phi_{n+1}). \quad (2)$$

The CP^n model is defined by the Lagrangian density \mathcal{L} in two-dimensional Euclidean space-time,

$$\mathcal{L} = \frac{1}{2g^2} [(D_\mu \Phi)(D_\mu \Phi)^\dagger + \alpha(\Phi \Phi^\dagger - 1)], \quad (3)$$

where D_μ ($\mu = 1, 2$) is the covariant derivative defined by

$$D_\mu = \partial_\mu + iA_\mu, \quad (4)$$

$$A_\mu = \frac{i}{2} [\Phi(\partial_\mu \Phi)^\dagger - (\partial_\mu \Phi)\Phi^\dagger], \quad (5)$$

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α is the multiplier field imposing the constraint $\Phi\Phi^\dagger=1$, and g is the coupling constant. This model has the global $SU(n+1)$ symmetry and the $U(1)$ local symmetry.

The CP^n model defined above has a topological charge Q , which is also called a winding number:

$$Q = \int d^2x q, \quad (6)$$

$$q = -\frac{1}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu = -\frac{i}{2\pi} \epsilon_{\mu\nu} (D_\mu \Phi)(D_\nu \Phi)^\dagger, \quad (7)$$

where q is the topological charge density.

The equation of motion is obtained from the Lagrangian density (3),

$$D_\mu D_\mu \Phi + (D_\mu \Phi)(D_\mu \Phi)^\dagger \Phi = 0. \quad (8)$$

If the self-dual equation

$$D_\mu \Phi = -i \epsilon_{\mu\nu} D_\nu \Phi \quad (9)$$

is satisfied, then the equation of motion is automatically satisfied. Using this self-dual equation, we get the topological charge density q as follows:

$$q = \frac{1}{2\pi} (D_\mu \Phi)(D_\mu \Phi)^\dagger, \quad (10)$$

which is proportional to the Lagrangian density under the constraint $\Phi^\dagger \Phi = 1$, namely, the relation between the Lagrangian density and the topological charge density is

$$\mathcal{L} = \frac{\pi}{g^2} q. \quad (11)$$

III. INFORMATION METRICS ON MODULI SPACES OF INSTANTONS

To find the self-dual solution, let us parametrize the field Φ as follows:

$$\Phi = \frac{W}{\sqrt{WW^\dagger}}, \quad (12)$$

where W is an $(n+1)$ -dimensional vector,

$$W = (w_1, w_2, \dots, w_{n+1}). \quad (13)$$

Substituting this Φ into the self-dual equation (9), we find that if $\partial_{\bar{z}} W = 0$, then the self-dual equation is satisfied, where $z = x^1 + ix^2$. Moreover, by making use of the $U(1)$ local symmetry, we can take W as $W = (\eta, u_2, u_3, \dots, u_{n+1})$, where η is a real number ($\eta \geq 0$) and u_i 's ($i=2,3,\dots,n+1$) are rational functions.

A. Information metric in the CP^1 model

Let us consider the instanton solution in the CP^1 model, which is the most simple $NL\sigma$ model. Since the purpose of

this paper is to investigate the information metric on the moduli space of the instanton with the topological charge which is greater than or equal to 1, we adopt the following self-dual solution:

$$W = [\lambda^k, (z-a)^k], \quad (14)$$

where $a = a^1 + ia^2$ and k is any positive integer. Here, λ and a correspond to instanton moduli parameters. The topological charge density of this instanton solution is given by

$$q = \frac{k^2}{\pi} \frac{\lambda^{2k} |z-a|^{2k-2}}{(\lambda^{2k} + |z-a|^{2k})^2}, \quad (15)$$

which has a maximum value where $|z-a| = [(k-1)/(k+1)]^{1/2k} \lambda$ ($k \geq 1$). The topological charge of the instanton solution (14) is k : $Q = k$. We take the normalized topological charge density, which corresponds to the normalized energy density or the normalized Lagrangian density in the Euclidean space-time, as the probability density function $p(z; \lambda, a)$ of the instanton in the CP^1 model:

$$p(z; \lambda, a) = \frac{1}{k} q = \frac{k}{\pi} \frac{\lambda^{2k} |z-a|^{2k-2}}{(\lambda^{2k} + |z-a|^{2k})^2}. \quad (16)$$

By substituting this probability density function into the definition of the information metric (1), we obtain

$$ds^2 = G_{AB} d\theta^A d\theta^B, \quad (17)$$

where

$$G_{11} = \frac{4}{3} k^2 \frac{1}{\lambda^2}, \quad (18)$$

$$G_{ij} = \frac{2\pi}{3} \frac{k^2 - 1}{k \sin(\pi/k)} \frac{1}{\lambda^2} \delta_{ij}, \quad (19)$$

$$G_{1i} = G_{i1} = 0, \quad (20)$$

where $\theta^1 = \lambda, \theta^2 = a^1, \theta^3 = a^2$ and $i, j = 2, 3$. If we change the instanton moduli parameter a into b as follows:

$$a = \sqrt{\frac{2}{\pi} \frac{k^3 \sin(\pi/k)}{k^2 - 1}} b, \quad (21)$$

where $\vec{b} = (b^1, b^2)$, it is established that the information metric on the moduli space (λ, \vec{b}) of the instanton is

$$ds^2 = \frac{4k^2}{3} \frac{d\lambda^2 + d\vec{b}^2}{\lambda^2}, \quad (22)$$

where $\vec{b} = (b^1, b^2)$. The overall scale factor of the information metric is $4k^2/3$. This hyperbolic three-space corresponds to the Euclidean anti-de Sitter space-time in three dimensions with the "radius" $R = \sqrt{4/3} k$, which means that the sectional curvature is $-3/4k^2$. If k becomes very large, the space-time becomes flatter. It is clear that the information

metric on the moduli space of one anti-instanton with the topological charge which is any negative integer is also the three-dimensional hyperbolic metric with the same curvature.

B. Information metric in the CP² model

In the CP² model, we consider the information metric on the moduli space of instantons which have $Q=1$ and moduli parameters (λ, a, b) . The instanton solution is

$$W = (\lambda, z - a, z - b), \tag{23}$$

where λ is real, a and b are complex numbers. In this case, the probability density function $p(z; \lambda, a, b)$ is given by

$$p(z; \lambda, a, b) = \frac{4}{\pi} \frac{2\lambda^2 + |a - b|^2}{(2\lambda^2 + |a - b|^2 + |2z - a - b|^2)^2}. \tag{24}$$

If we change the coordinate z and the moduli parameters (λ, a, b) into v and (Λ, c, δ) as follows:

$$z = \frac{1}{\sqrt{2}}v, \tag{25}$$

$$a = \frac{1}{\sqrt{2}}(c + \delta), \tag{26}$$

$$b = \frac{1}{\sqrt{2}}(c - \delta), \tag{27}$$

$$\lambda = \sqrt{\Lambda^2 - \delta^2}, \tag{28}$$

the probability density function becomes

$$\tilde{p}(v; \Lambda, c, \delta) = \frac{1}{\pi} \frac{\Lambda^2}{(\Lambda^2 + |v - c|^2)^2}. \tag{29}$$

This is indeed the same as the probability density function for one instanton with $Q=1$ in the CP¹ model [see Eq. (16)]. Therefore, the information metric of the moduli space (Λ, c, δ) is easily obtained,

$$ds^2 = \frac{4}{3} \frac{d\Lambda^2 + d\vec{c}^2}{\Lambda^2}, \tag{30}$$

where $\vec{c} = (c^1, c^2)$. This information metric represents the hyperbolic three-space, which corresponds to the Euclidean anti-de Sitter space-time in three dimensions, in the same way as the case of the CP¹ model mentioned in Sec. III A. Notice that there is no dependence on the parameters δ and $d\vec{\delta}$ in this information metric, where $\vec{\delta} = (\delta^1, \delta^2)$. The substitution of Eqs. (25)–(28) into Eq. (30) leads to the information metric for the moduli parameters (λ, a, b) :

$$ds^2 = G_{AB} d\theta^A d\theta^B, \tag{31}$$

$$G_{11} = \frac{16}{3} \frac{\lambda^2}{(2\lambda^2 + |a - b|^2)^2}, \tag{32}$$

$$G_{1i} = G_{i1} = -G_{1i+2} = -G_{i+2\ 1} = \frac{8}{3} \frac{\lambda \tilde{\delta}_i}{(2\lambda^2 + |a - b|^2)^2}, \tag{33}$$

$$G_{ij} = G_{ji} = G_{i+2\ j+2} = G_{j+2\ i+2} = \frac{4}{3} \frac{\tilde{\delta}_i \tilde{\delta}_j}{(2\lambda^2 + |a - b|^2)^2} + \frac{4}{3} \frac{\delta_{ij}}{2\lambda^2 + |a - b|^2}, \tag{34}$$

$$G_{ij+2} = G_{j+2\ i} = -\frac{4}{3} \frac{\tilde{\delta}_i \tilde{\delta}_j}{(2\lambda^2 + |a - b|^2)^2} + \frac{4}{3} \frac{\delta_{ij}}{2\lambda^2 + |a - b|^2}, \tag{35}$$

where $\theta^1 = \lambda$, $\theta^2 = a^1$, $\theta^3 = a^2$, $\theta^4 = b^1$, $\theta^5 = b^2$, $i, j = 2, 3$, $A, B = 1, 2, \dots, 5$, and $\tilde{\delta}_i = a^i - b^i$. Although this information metric looks complicated, a simple structure is hidden as explained above.

IV. CONCLUSIONS AND DISCUSSIONS

We have shown that in the CP¹ model the information metric on the moduli space of one instanton with the topological charge $Q=k$ ($k \geq 1$) is the Euclidean anti-de Sitter space-time metric in three dimensions. The overall scale factor of the information metric is $4k^2/3$, and this means that the ‘‘radius’’ $R = \sqrt{4/3}k$ and the sectional curvature is $-3/4k^2$. If k becomes very large, the space-time becomes flatter. Furthermore, we have also computed the information metric of the moduli space of the instanton in the CP² model.

The NL σ model often arises in string theory as well as in various statistical mechanics problems. Since the topological charge of the instanton may be related to the Ramond-Ramond charge or the D -instanton charge in string theory, it is worthwhile to attempt to apply the above results to the correspondence between conformal field theories and string theories in the anti-de Sitter space-time, which is called the AdS/CFT correspondence [22]. It is particularly interesting to investigate the relation between three-dimensional anti-de Sitter space-time and a kind of two-dimensional NL σ model from the point of view of string theory. Furthermore, we should study the information metric on the moduli space of multiinstantons in the NL σ models, the YM theories, and supergravity theory in more detail.

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